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A new method for solving the inverse conduction problem in steady heat flux measurement

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Abstract—Based on the linear superposition theorem a procedure is proposed for determining steady heat fluxes on boundaries which pose an inverse conduction problem. Error analysis indicates that the condition number of the resulting coefficient matrix is the decisive factor for successful application of this approach. A practical case is used to demonstrate application of the method for heat flux measurement on a membrane wall assembly. The new method has advantages in terms of simplicity and flexibility. © 1997 Elsevier Science Ltd.

INVERSE HEAT CONDUCTION PROBLEMS

Many problems in science and engineering including heat transfer problems are described by ordinary or partial differential equations. For example, if appropriate boundary conditions (and, for transient problems, an initial condition) together with all the key variables (thermal conductivity, density, specific heat, etc.) are known, the governing heat conduction equation for a solid body can be solved analytically or numerically. The temperature distribution is then found. This is termed a direct problem. However, if any of the above conditions, such as the heat flux on the solid surfaces, its physical properties or even its geometrical size or shape must be determined from temperature measurement at certain locations, this represents an inverse problem. Two general classes of the inverse conduction problems commonly encountered are the parameter estimation problem and the function estimation problem [1]. The former tries to determine parameters in the heat conduction problem such as the thermophysical properties or the heat transfer coefficient on the boundary based on a measured temperature distribution or response. A typical example of the function estimation is to find the transient heat flux on the boundary (as a function of time) on the basis of the temperature response of an object. The solution of inverse problems is much more difficult than that of direct problems. Since inverse heat conduction includes a substantial variety of problems, their solution requires a variety of strategies. The following discussion focuses on application of inverse problems in estimating one or more steady heat fluxes on a surface by means of temperature measurement. Mathematically, such a problem may also be classified as a parameter estimation problem.

Inverse conduction problems can also be classified as deterministic or nondeterministic, also called nonstochastic or stochastic. Based on transient conduction analysis some test programs have been developed to determine multiple properties, which involve either deterministic or nondeterministic parameter estimation [1–3]. Nondeterministic parameter estimation usually involves nonlinear optimization algorithms. The merit function then usually has a number of local minima. Algorithms have been devised for nonlinear multidimensional minimization to search for a local minimum, starting from some approximate trial solution. Little is known about finding global extremes in general cases. Success often depends on having a good first-guess for the solution. On the contrary, deterministic inverse problems can lead to a definite solution instead of an "estimation". This kind of approach is useful in practice, especially in measurements. Here we discuss a special class of inverse conduction problem used to determine steady heat fluxes on solid surfaces.

THEORETICAL BASIS

Consider a generalized steady-state heat conduction problem with an arbitrary geometry and, possibly,

NOMENCLATURE

A	coefficient matrix [m^2 K W^{-1}]
d	right-side vector [°C]
h	heat transfer coefficient [W m^{-2} K^{-1}]
m	number of unknown fluxes
n	vector normal to boundary [m]
$\mathbf{P}_j$	coordinate of temperature sensing location [m]
q	heat flux vector [W m^{-2}]
q	heat flux to be determined [W m^{-2}]
q_v	internal heat source intensity [W m^{-3}]
t	temperature [°C]
t_c	temperature component [°C]
T	measured temperature [°C]
T_a	ambient temperature [°C]
T_f	coolant temperature [°C].

Greek symbols

λ	thermal conductivity [W $(mK)^{-1}$]
θ	defined in equation (2) [m^2 K W^{-1}]
ϕ	defined in equation (11).

Subscripts

b	boundary
i, j, k	integers
s	surface.

with a heterogeneous internal heat source. As shown in Fig. 1, the boundary of the studied domain can be divided into several sections. Among them m sections of the boundaries are exposed to unknown, but uniform, heat fluxes, q_i ($i = 1, 2, \ldots, m$), respectively. Other sections (b_1, b_2, b_3) are subject to fixed temperature, flux or convective boundary conditions. The conduction problem can then be described by the following mathematical formulation.

$$\left.\begin{aligned} &\nabla\cdot\lambda\nabla t + q_v = 0 \\ &t|_{b1} = t_b(s) \\ &\lambda\frac{\partial t}{\partial n}\bigg|_{b2} = q_b(s) \\ &-\lambda\frac{\partial t}{\partial n}\bigg|_{b3} = h(t|_{b3} - t_f) \\ &\lambda\frac{\partial t}{\partial n}\bigg|_{si} = q_i, \quad (i = 1, 2, \ldots, m) \end{aligned}\right\}. \tag{1}$$

Note that the above formulation is composed solely of linear expressions for the case with λ taken as a constant property. In the light of the superposition theorem, the temperature distribution defined by the above formulation can be expressed as the sum of solutions of several simpler linear problems:

$$\begin{aligned} t &= t_c + \sum_{i=1}^{m} t_i \\ &= t_c + \sum_{i=1}^{m} q_i\theta_i. \end{aligned} \tag{2}$$

Here t_c denotes the basic temperature distribution determined by the internal heat source and other boundary conditions on b_1, b_2, b_3 when $q_i = 0$ ($i = 1, 2, \ldots, m$); t_i is a temperature supplement caused uniquely by heat flux q_i on boundary section s_i when the governing equation and all other boundary conditions are homogeneous. That is

$$\left.\begin{aligned} &\lambda\nabla\cdot\nabla t_c + q_v = 0 \\ &t_c|_{b1} = t_b(s) \\ &\lambda\frac{\partial t_c}{\partial n}\bigg|_{b2} = q_b(s) \\ &-\lambda\frac{\partial t_c}{\partial n}\bigg|_{b3} = h(t_c|_{b3} - t_f) \\ &\lambda\frac{\partial t_c}{\partial n}\bigg|_{si} = 0, \quad (i = 1, 2, \ldots, m) \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} &\nabla\cdot\nabla t_i = 0 \\ &t_i|_{b1} = 0 \\ &\lambda\frac{\partial t_i}{\partial n}\bigg|_{b2} = 0 \\ &-\lambda\frac{\partial t_i}{\partial n}\bigg|_{b3} = ht_i|_{b3} \\ &\lambda\frac{\partial t_i}{\partial n}\bigg|_{sk} = \begin{cases} q_k, & k = i \\ 0, & k \neq i \end{cases} \quad (k = 1, 2, \ldots, m) \end{aligned}\right\}$$

$$(i = 1, 2, \ldots, m). \tag{4}$$

θ_i in equation (2) is numerically equal to t_i for $q_i = 1$.

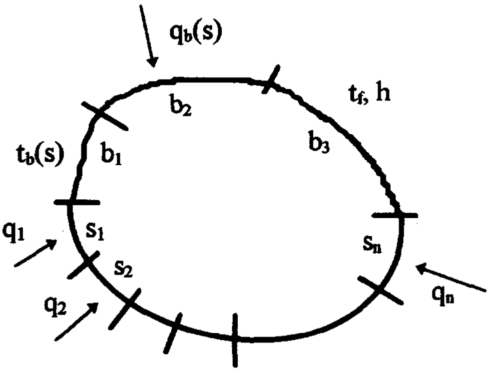

Fig. 1. Schematic of boundary of heat conduction system on which boundary conditions are applied.

As a result, θ_i is determinate for a specified problem, although the heat fluxes on the boundary, q_i, are variable and to be determined.

There are a number of mature methods to solve conduction problems, including various analytical and numerical approaches. The finite difference method (FDM) and finite element method (FEM) are powerful commonly-used techniques for conduction problems with irregular geometry, complex boundary conditions and/or variable properties. Consequently, it is possible to solve equations (3) and (4) and determine t_c and θ_i as functions of coordinates providing that the configuration and thermal parameters are specified. The final solution for the temperature distribution depends also upon the unknown heat fluxes on the boundary, q_i. Provided the temperatures T_j at different locations P_j $(j = 1, 2, \ldots, m)$ can be measured, we obtain

$$T_j = t_c(P_j) + \sum_{i=1}^{m} q_i \theta_i(P_j), \quad (j = 1, 2, \ldots, m). \quad (5)$$

With the aid of matrix notation $\mathbf{A} = [\theta_i(P_j)]$, $\mathbf{q} = [q_1, q_2, \ldots, q_m]^T$ and $\mathbf{d} = [d_1, d_2, \ldots, d_m]^T$, the above linear equations can be rewritten as

$$\mathbf{Aq} = \mathbf{d} \quad (6)$$

or

$$\mathbf{q} = \mathbf{A}^{-1}\mathbf{d} \quad (7)$$

where $d_j = T_j - t_c(P_j)$. The unknown heat fluxes can be found if this set of linear algebraic equations can be solved.

ERROR ANALYSIS

It is necessary to assess the credibility, or accuracy of this approach in determining heat fluxes on solid surfaces. Firstly, the geometrical configuration and relevant parameters of the test rig need to be optimized for practical measurements. When the configuration is chosen, various options exist as to where to install the temperature sensors. Dealing with the positioning properly can be crucial for successful measurement.

The set of algebraic equations, expression (6), is fully determined by its coefficient matrix $\mathbf{A}$ and the vector $\mathbf{d}$. They serve as original data in solving the equations. The coefficient matrix $\mathbf{A}$ is computed by solving equation (4). As a result of model inaccuracy and parameter indeterminacy and also because of algorithm errors and round-off errors in computation, it is inevitable that coefficient matrix $\mathbf{A}$ possesses certain errors. Similarly, the right-side vector $\mathbf{d}$ cannot be precise; it is obtained through temperature measurement, as well as by solving equation (3). These errors cause indeterminacy of the final results.

Suppose, firstly, that the coefficient matrix $\mathbf{A}$ is accurate, but the right-side vector $\mathbf{d}$ is not. The right-side of the equation is then designated as $\mathbf{d} + \delta\mathbf{d}$, which corresponds to a solution $\mathbf{q} + \delta\mathbf{q}$. Assuming $\mathbf{d} \neq 0$ (and hence $\mathbf{q} \neq 0$), it may be proved [4, 5] that

$$\frac{\|\delta\mathbf{q}\|}{\|\mathbf{q}\|} \leqslant \|\mathbf{A}\| \, \|\mathbf{A}^{-1}\| \frac{\|\delta\mathbf{d}\|}{\|\mathbf{d}\|}. \quad (8)$$

Alternatively, suppose that $\mathbf{d}$ is accurate while $\mathbf{A}$ is subject to error and becomes $\mathbf{A} + \delta\mathbf{A}$. The corresponding solution changes to $\mathbf{q} + \delta\mathbf{q}$. A similar relation is then obtained:

$$\frac{\|\delta\mathbf{q}\|}{\|\mathbf{q} + \delta\mathbf{q}\|} \leqslant \|\mathbf{A}\| \, \|\mathbf{A}^{-1}\| \frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|}. \quad (9)$$

$\|\mathbf{A}\| \, \|\mathbf{A}^{-1}\|$ is referred to as the condition number of the non-singular $\mathbf{A}$ and designated *cond*($\mathbf{A}$). The character of the condition number requires that *cond*($\mathbf{A}$) $\geqslant 1$. Equations (8) and (9) indicate that it is *cond*($\mathbf{A}$) that decides the relative error transfer from the original data to the solution of the linear algebraic equations. If the condition number is large enough, indicating that the set of equations is 'ill-conditioned', the indeterminacy of the original data makes a strong impact on the solution.

As a consequence, when a test scheme is designed for heat flux measurement by the method discussed above, it is important to determine first the condition number of the coefficient matrix in order to verify the feasibility of the scheme. Hence, also when the locations are to be decided for temperature sensors, the vicinity where θ_i reaches its extreme should be chosen as the ith temperature sensing point. The rationality of the overall arrangement of temperature sensing points should be further tested using *cond*($\mathbf{A}$).

SAMPLE APPLICATION

Heat flux measurement for membrane walls

The method is illustrated by considering heat flux measurement in membrane waterwalls. Consisting of parallel tubes connected longitudinally by fins, membrane waterwalls are insulated on one side and exposed to a furnace on the other, as shown schematically in Fig. 2. These walls provide an effective means of transferring heat from a furnace to water being heated or boiled on the inside of the tubes, and have been widely used in both conventional and circulating fluidized bed (CFB) boilers. It is vital to be able to characterize the heat flux on the membrane walls both on the laboratory study of heat transfer

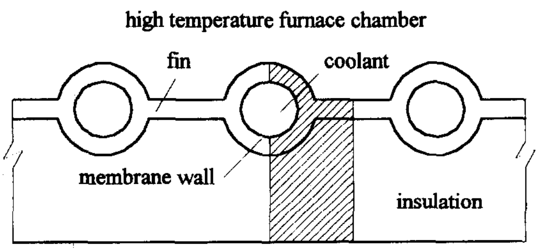

Fig. 2. Membrane wall assembly representative wall section showing portion (shaded) treated in the analysis.

and in industrial applications. To measure heat flux on membrane walls with accuracy, however, is a challenging task. Considerable work has been done in recent years in this field [6, 7]. By means of both analytical and numerical solutions, Bowen *et al.* [8] analyzed the temperature distribution in membrane walls perfectly insulated on their outer side for uniform local furnace-side heat transfer coefficients. Taler's method [9] involves the solution of a set of nonlinear algebraic equations. In the mathematical sense and in practice it is likely that the nonlinear equations are unable to be satisfied, i.e. roots do not exist.

Taking advantage of the linear conduction system, the method proposed here features simplicity and flexibility and allows disturbances to be minimized. For the purpose of heat flux measurement, a heat conduction model of the membrane waterwall and the insulation further assumes that:

(1) the heat conduction is at steady-state;

(2) the temperature gradient along the height of the wall is much smaller than that in its cross-section, so that the conduction can be approximated as being two-dimensional;

(3) the properties of the wall and the insulation are isotropic and independent of temperature;

(4) the coefficient of heat transfer between the tube and the coolant, h_f, is uniform over the inner tube surface;

(5) the coefficient of heat transfer between the insulation and the ambient exterior of the furnace, h_a, is also uniform.

Because of the geometric complexity and the different properties of the membrane waterwall and insulation, a rigorous analysis of the waterwall conduction problem requires use of numerical methods. The solution is obtained here by employing the finite element technique.

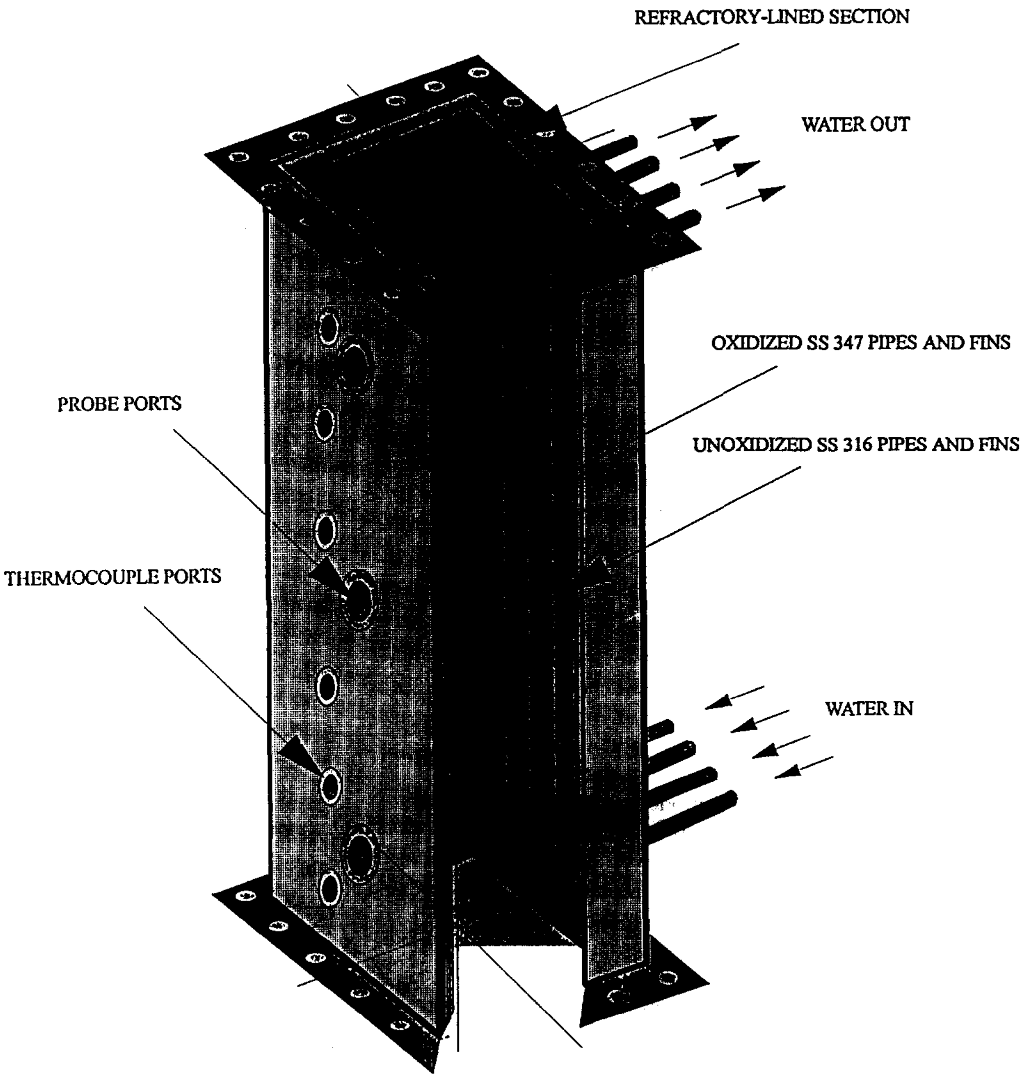

Fig. 3. Test section equipped with membrane wall for CFB heat transfer study.

Test section layout

A test membrane wall assembly has been constructed at The University of British Columbia for laboratory study of circulating fluidized bed heat transfer, as illustrated in Fig. 3. Because of symmetry considerations, only the area of the hemi-cylinder, half-fin and the insulation of the test assembly needs to be analyzed. Figure 4 shows the physical dimensions of the analyzed domain. The parameters are as follows: $R_i = 7.05$ mm, $R_0 = 10.65$ mm, $S = 17.05$ mm, $H = 103$ mm, $B = 4.8$ mm, $\lambda_m = 16$ W $(\text{mK})^{-1}$, $\lambda_i = 0.15$ W $(\text{mK})^{-1}$. In order to understand the heat flux variation on the tube and the fin, the furnace-side boundary is divided into three sections as shown in Fig. 4. Section 1 covers an angle of 52° of the tube. The heat fluxes, q_1, q_2, q_3 differ from each other, and each is supposed to be uniform, as required by the model. Temperatures T_1, T_2, T_3 are measured at the tube-crest top, the center of the outside surface of the second section and the center of the fin, as indicated by P_1, P_2, P_3 in Fig. 4. The mean temperature of the water at different levels and the ambient temperature are also measured. The heat transfer coefficients to cooling water inside the tube and to ambient air outside the insulation are obtained using classical correlations from the literature [10]. For an experiment at low temperature ($T_f = 30.3$°C, $T_a = 20.0$°C) and low flow rate, for instance, we obtained $h_f = 2800$ W m^{-2} K^{-1} and $h_a = 40$ W m^{-2} K^{-1}.

Computational results

Equations (2)–(4) as well as the parameters given above allow the components θ_1, θ_2, θ_3 and t_c to be computed. Again, taking the advantage of linearity, a factor ϕ is introduced to indicate the influence of the ambient temperature on the heat loss through the insulation. Numerically, ϕ is the distribution of temperature excess over T_f when $T_a - T_f = 1$°C. That is

$$t = t_c + \sum_{i=1}^{3} q_i\theta_i$$

$$= T_f + (T_a - T_f)\phi + \sum_{i=1}^{3} q_i\theta_i. \quad (10)$$

Figure 5 shows the temperature components θ_1, θ_2, θ_3 and ϕ. A typical superposed temperature field is plotted in Fig. 6 for $q_1 = 1.0 \times 10^5$, $q_2 = 1.1 \times 10^5$, $q_3 = 1.2 \times 10^5$ W m^{-2}. The temperatures at the three sensing points are then obtained as $T_1 = 105.42$°C, $T_2 = 111.17$°C and $T_3 = 134.61$°C using the simulation. The temperatures at the three selected sensing points are functions of q_1, q_2 and q_3. For the above-specified conditions:

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 30.30 \\ 30.29 \\ 30.28 \end{bmatrix} + \begin{bmatrix} 58.63 & 8.01 & 6.40 \\ 22.71 & 27.80 & 22.99 \\ 12.57 & 14.39 & 63.28 \end{bmatrix} \times 10^{-5} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}. \quad (11)$$

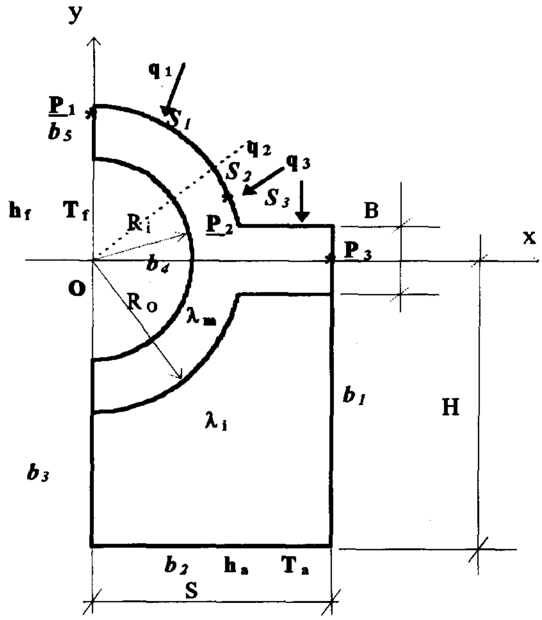

Fig. 4. Boundary conditions for the membrane wall conduction system. Thermocouples are located at points P_1, P_2 and P_3.

Let

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} T_1 - 30.30 \\ T_2 - 30.29 \\ T_3 - 30.28 \end{bmatrix}. \quad (12)$$

Solving equation (11), we obtain

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1.9198 & -0.5578 & 0.0086 \\ -1.5436 & 4.8798 & -1.6172 \\ -0.0302 & -0.9993 & 1.9465 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \times 10^3 \text{ W m}^{-2} \quad (13)$$

Error estimation

From equation (11) the condition number of its coefficient matrix can be calculated, giving *cond*(**A**) = 4.6828. Assume that the errors in the temperature measurement are ± 0.5°C. In view of the very minor influence of heat loss through the insulation, the maximum absolute error in the temperature differences, d_i, may be estimated to be ± 1°C. Then we have

$$\frac{\|\delta\mathbf{d}\|}{\|\mathbf{d}\|} = \frac{\sqrt{1^2+1^2+1^2}}{\sqrt{(105.42-30.30)^2+(111.17-30.29)^2+(134.61-30.28)^2}} = 0.0114.$$

According to equation (8) the relative errors in **q** would be limited by

$$\frac{\|\delta\mathbf{q}\|}{\|\mathbf{q}\|} \leqslant \text{cond}(\mathbf{A})\frac{\|\delta\mathbf{d}\|}{\|\mathbf{d}\|} = 4.6828 \times 0.0114 = 0.0534.$$

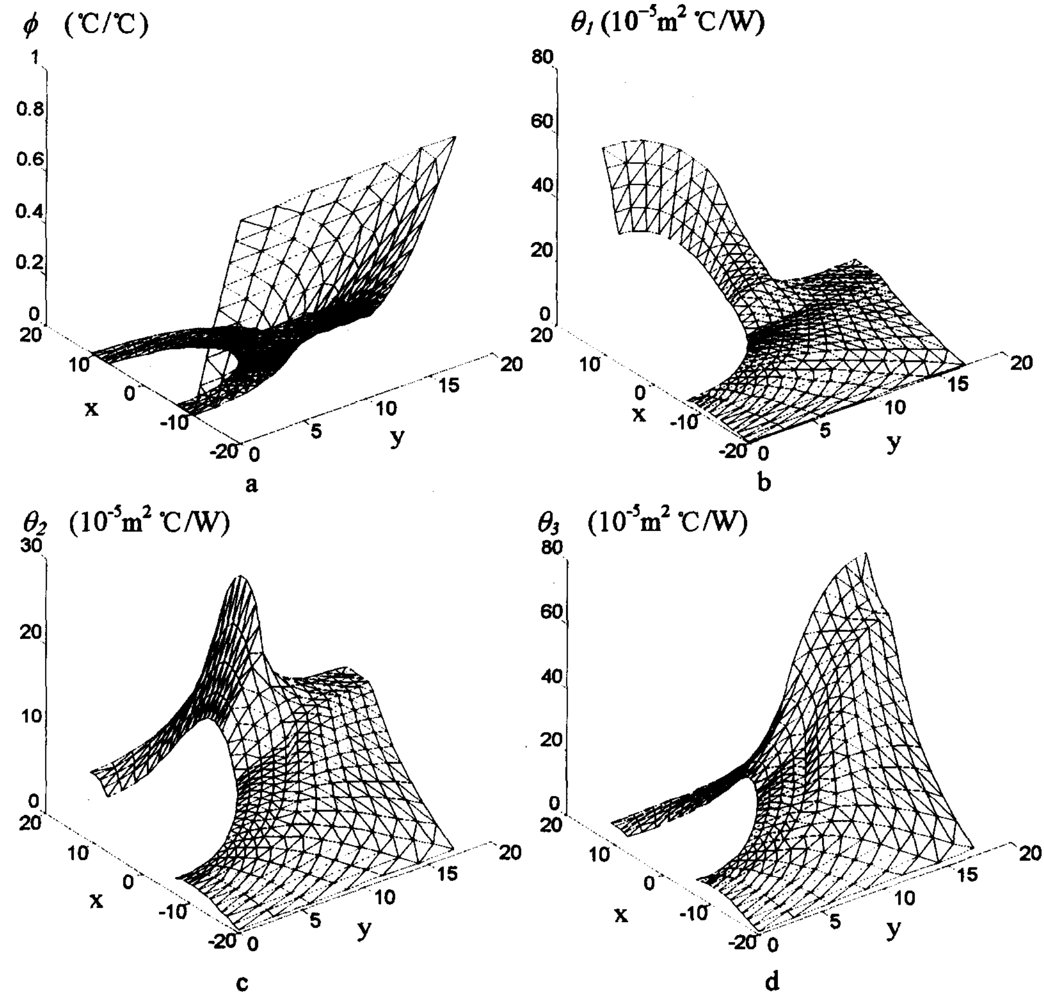

Fig. 5. Distribution of temperature components a: ϕ, b: θ_1, c: θ_2, d: θ_3 [$R_i = 7.05$ mm, $R_0 = 10.65$ mm, $S = 17.05$ mm, $H = 20$ mm, $B = 4.8$ mm, $h_f = 2800$ W m^{-2} K^{-1}, $h_a = 40$ W m^{-2} K^{-1}, $\lambda_i = 0.15$ W (mK)$^{-1}$].

For simulation of the error transfer, an increment of 1°C was assigned to the computed precise temperatures T_1, T_2 and T_3, respectively. The simulated heat fluxes were evaluated using equation (13), and their relative errors were estimated.

$$\mathbf{q} = |1.014 \quad 1.117 \quad 1.212|^{T} \times 10^5 \text{ W m}^{-2},$$

so that $\delta\mathbf{q} = |0.014 \quad 0.017 \quad 0.012|^{T} \times 10^5$ W m^{-2},

$$\frac{\|\delta\mathbf{q}\|}{\|\mathbf{q}\|} = \frac{\sqrt{0.014^2+0.017^2+0.012^2}}{\sqrt{1.0^2+1.1^2+1.2^2}} = 0.0172 < 0.0534.$$

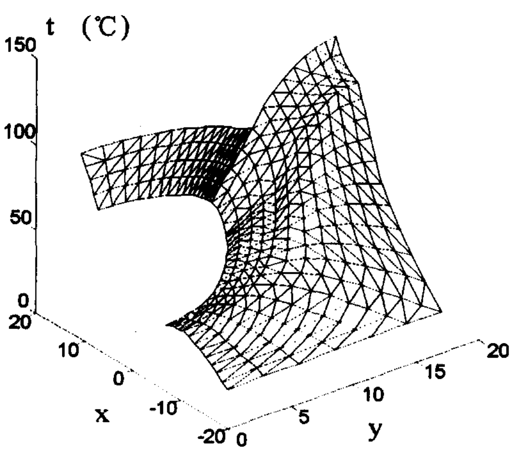

Fig. 6. Typical superposed temperature field [$q_1 = 1.0 \times 10^5$, $q_2 = 1.1 \times 10^5$, $q_3 = 1.2 \times 10^5$ W m^{-2}].

CONCLUDING REMARKS

Taking advantage of the linearity of the system of equations, the proposed approach for solving the inverse heat conduction problems features simplicity and determinacy. Both are of great significance for measurements. The scheme is illustrated in practice by treating a case where we wish to determine the heat fluxes on membrane waterwalls. The feasibility of the method for this real application is demonstrated using error analysis. Further study is needed to investigate other influences such as longitudinal gradients, errors in the heat transfer coefficient and geometric inaccuracies in the membrane wall example.

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